A Translation-Invariant Geometric–Arithmetic Framework for the Natural Numbers

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Abstract

We introduce a class of translation- and scale-invariant geometric–arithmetic structures that encode divisibility, factorization, and primality of the integers as positional and intersection properties. The framework is based on a finite periodic decomposition, a family of generator cycles, and a notion of structural intersection. Prime numbers arise as isolated positions on admissible arithmetic rays, while composite numbers correspond to nontrivial intersections of generating cycles. The construction extends symmetrically to all integers and admits both geometric and algebraic realizations. We formalize these structures categorically and show that they are equivalent to classical arithmetic while providing a unified structural perspective underlying divisibility, Euler's product, and the global organization of the integers.

1. Introduction

Classical number theory studies the natural numbers primarily through algebraic and analytic methods, with primality defined in terms of divisibility or algorithmic tests. Alongside this tradition, various geometric and structural representations—lattices, divisibility graphs, sieves, and spectral analogies—have provided alternative viewpoints without altering the underlying arithmetic content.

The purpose of this paper is not to obtain new estimates or distributional theorems for prime numbers. Instead, we introduce and formalize a **class of geometric–arithmetic invariant structures** in which the arithmetic of the integers is encoded as a positional and intersectional geometry. Within this framework:

- divisibility becomes alignment within periodic generator cycles;
- composite numbers correspond to structural intersections;
- prime numbers appear as isolated admissible positions;
- the global structure is invariant under translation and scaling.

The framework admits both geometric and algebraic constructions and yields a categorical organization of arithmetic structure.

2. The Integers as a Translation-Invariant Geometric Domain

We consider the set of integers

$$\mathbb{Z}$$

as an infinite discrete geometric lattice equipped with a translation-invariant metric

$$d(n,m) = \mid n - m \mid.$$

Axiom 2.1 (Reference invariance)

All structural properties of the framework are invariant under translations of the form

$$\tau_k(n) = n + k, k \in \mathbb{Z}.$$

Thus, no integer plays a privileged role; any element may serve as a reference point.

3. Periodic Decomposition and the Cycle of Order Six

A fundamental structural feature is the finite periodic decomposition induced by congruence modulo six.

Definition 3.1 (Cycle-6 decomposition)

For a chosen reference point $r \in \mathbb{Z}$, define translated coordinates

$$n_r = n - r,$$

and the congruence map

$$\phi(n_r) = n_r \, mod 6.$$

This partitions $\mathbb{Z}$into six infinite congruence classes.

Proposition 3.2

For any integer $n$with $| n_r |> 3$,

$$\phi(n_r) \notin \{1,5\} \Rightarrow n \text{ is composite.}$$

This classical arithmetic fact is here interpreted structurally: only two congruence classes remain admissible for primality beyond trivial cases.

4. Admissible Rays and Symmetry

Definition 4.1 (Admissible set)

The admissible positions relative to $r$are

$$\mathcal{A}_r = \{n \in \mathbb{Z}: | n_r |> 1, \; n_r \equiv \pm 1 \pmod 6\}.$$

This set decomposes into two infinite arithmetic rays.

Proposition 4.2 (Reflection symmetry)

If $n \in \mathcal{A}_r$, then $2r - n \in \mathcal{A}_r$.

Hence the framework extends symmetrically to negative integers, with primality invariant under reflection.

5. Generator Cycles and Structural Intersections

Definition 5.1 (Generator cycles)

For each integer $a$with $| a |> 1$, define the cycle

$$C_a = \{ak + r : k \in \mathbb{Z}\}.$$

These cycles encode divisibility relations.

Definition 5.2 (Structural intersection)

An integer $n$admits a **nontrivial structural intersection** if there exist distinct integers $a, b$with

$$1 <| a |, | b |<| n_r |$$

such that

$$n \in C_a \cap C_b.$$

6. Primality as Structural Isolation

Theorem 6.1

An integer $n \in \mathcal{A}_r$is composite if and only if it admits a nontrivial structural intersection.

Definition 6.2 (Geometric primality)

An integer $p \in \mathcal{A}_r$is **geometrically prime** if it admits no nontrivial structural intersection.

Theorem 6.3 (Equivalence)

Geometric primality is equivalent to classical primality.

Thus, primes are characterized as isolated positions within a finite periodic geometric framework.

7. Scale Invariance and Infinitude

Proposition 7.1

The configuration of admissible rays and structural intersections is invariant under scaling of the lattice.

Theorem 7.2

The infinitude of prime numbers follows from the infinite extent of admissible rays together with the scale invariance of the isolation criterion.

This argument relies on structural persistence rather than limiting computation.

8. Algebraic–Geometric Duality and Euler’s Product

The same framework admits an algebraic realization: generator cycles correspond to arithmetic progressions determined by divisibility, while structural intersections encode factorization.

From this perspective:

- prime numbers act as irreducible generators;
- composite numbers arise from overlaps of generator cycles;
- Euler's product formula reflects the decomposition of global arithmetic structure into independent prime components.

This establishes a dual geometric–algebraic description without altering classical results.

9. Categorical Formulation

We formalize these structures as a category **CGAI** (Category of Geometric–Arithmetic Invariant structures), whose objects encode the data above and whose morphisms are global translations preserving all structure.

Proposition 9.1

CGAI is a groupoid in which all objects are isomorphic.

Thus, arithmetic structure is independent of the observer's reference point.

10. Delimitation of Contributions

- **Results:** structural characterization of primality via intersections; translation and scale invariance; categorical formulation.
- **Reformulations:** primes as isolated positions; composites as intersections; factorization as cycle decomposition.
- **Interpretations:** resonance, interference, and wave-like analogies are heuristic and not used in proofs.

11. Conclusions

We have introduced a natural class of geometric–arithmetic invariant structures that encode the arithmetic of the integers in a finite, translation-invariant, and scale-invariant manner. The framework is equivalent to classical number theory yet reorganizes it around geometry, symmetry, and structure rather than computation. While no new distributional results are claimed, the approach provides a unified structural perspective on divisibility, primality, factorization, and Euler's product.

12. Aspects of Novelty and Original Contributions

We explicitly enumerate below the novel aspects of the present work, distinguishing them from classical results and standard reformulations.

12.1. A translation-invariant axiomatic framework for arithmetic structure

The paper introduces a **fully translation-invariant axiomatic framework** in which all arithmetic properties—divisibility, factorization, and primality—are defined relative to an arbitrary reference point. While translation invariance is implicit in classical arithmetic, it is here made **structural and explicit**, with all definitions formulated in reference-free terms.

This yields a setting in which no integer plays a privileged role, and arithmetic laws emerge as properties of the global structure rather than of specific numerical values.

12.2. Primality as structural isolation rather than divisibility testing

A central novelty is the **reformulation of primality as a geometric isolation property** within a finite periodic framework. Prime numbers are characterized by the **absence of nontrivial structural intersections** among generator cycles.

This contrasts with classical definitions, which identify primes via universal divisibility conditions or algorithmic tests. Here, primality is detected structurally and finitely, without iteration or limiting procedures.

12.3. Finite periodic control of an infinite structure

The framework shows that an **infinite arithmetic object**—the set of all primes—can be governed by a **finite periodic decomposition** (of order six) combined with an intersection criterion.

While congruence conditions modulo small integers are classical, their integration into a **complete geometric characterization** of primes, composites, and factorization is new.

12.4. Symmetric extension to negative integers

The model treats positive and negative integers on equal footing. Primality, factorization, and divisibility are shown to be **invariant under reflection** across any reference point.

This symmetric extension is not merely formal but structural, reinforcing the interpretation of numbers as positions in a geometric lattice rather than as ordered magnitudes.

12.5. Dual geometric–algebraic realizability

The same structure admits:

- a **geometric realization**, via positions, cycles, distances, and intersections;
- an **algebraic realization**, via congruences, arithmetic progressions, and divisibility.

This establishes a precise **duality between geometric and algebraic generation** of arithmetic structure, without introducing new axioms beyond the integers themselves.

12.6. Categorical organization of arithmetic invariance

The introduction of the category **CGAI** provides a novel categorical formalization of arithmetic invariance. Objects represent complete arithmetic universes, while morphisms correspond to changes of reference.

This categorical viewpoint isolates the **observer-independence** of arithmetic structure and formalizes it in standard categorical language.

12.7. Structural reinterpretation of Euler's product

Without altering analytic number theory, the framework offers a **structural reinterpretation** of Euler's product as a decomposition of global arithmetic structure into independent generator cycles.

This provides a geometric–structural intuition underlying the classical analytic identity, while remaining strictly equivalent to it.

13. Scope and Limitations

We emphasize that the present work:

- does **not** claim new bounds, density results, or analytic estimates for primes;
- does **not** modify classical arithmetic truths;
- introduces no probabilistic or computational assumptions.

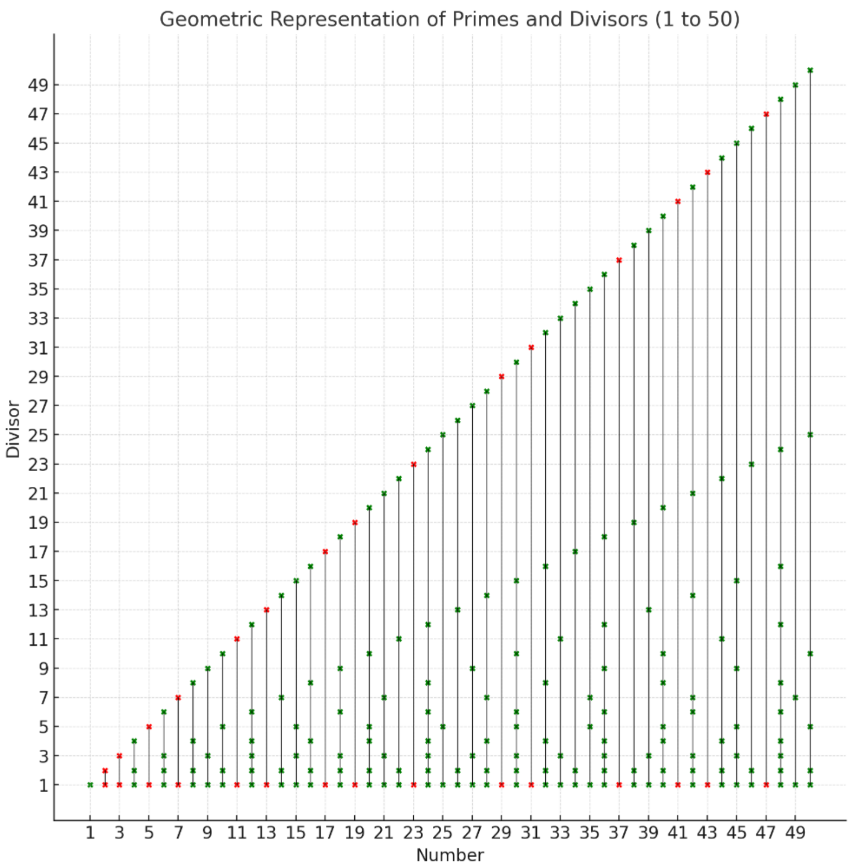
Its contribution lies in **structure, organization, and invariance**, rather than in quantitative novelty.

14. Concluding Perspective

The framework presented here identifies a natural class of geometric–arithmetic structures underlying the integers, within which primes, composites, and factorizations arise as positional and intersectional phenomena. By making invariance, symmetry, and periodicity explicit, the model provides a unified structural perspective on classical arithmetic and clarifies the relationship between geometric intuition and algebraic formalism.

References

Euclid, *Elements*, Book IX
Hardy, G. H.; Wright, E. M., *An Introduction to the Theory of Numbers*
Nathanson, M. B., *Elementary Methods in Number Theory*



A.

A.

B.

1	2	3	4	5	6	7	8	9	10	11	12	13
	1		2		3		4		5		6	
		1			2			3			4	
			1				2				3	
				1					2			
					1						2	
						1						
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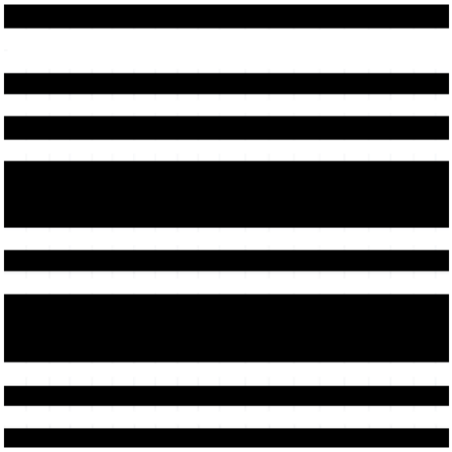
C.



D.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0	0	0	0	0	0
4	1	1	0	1	0	0	0	0	0	0	0	0	0
5	1	0	0	0	1	0	0	0	0	0	0	0	0
6	1	1	1	0	0	1	0	0	0	0	0	0	0
7	1	0	0	0	0	0	1	0	0	0	0	0	0
8	1	1	0	1	0	0	0	1	0	0	0	0	0
9	1	0	1	0	0	0	0	0	1	0	0	0	0
10	1	1	0	0	1	0	0	0	0	1	0	0	0
11	1	0	0	0	0	0	0	0	0	0	1	0	0
12	1	1	1	1	0	1	0	0	0	0	0	1	0
13	1	0	0	0	0	0	0	0	0	0	0	0	1

Examples of frames Natural matrices based on indicating the divisibility of numbers. These can be constructed by geometric means without algebraic calculation, or using mathematical calculation methods (simultaneous geometric-algebraic dualism)